Meriam 6/3 – 6/5 : Kinetics of Planar Bodies

Translation:

For 1 particle : $\sum \vec{F} = m\vec{a}$

For N particles : $\sum \vec{F} = m\vec{a}_G$ (also written as $m\bar{a}$)

where $m = \sum_{i=1}^{N} m_i$

$$\vec{a}_G = \dot{\vec{V}} = \ddot{\vec{r}}$$

$$\vec{r}_G = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{m}$$

But a solid body is just a very large number of particles glued together.

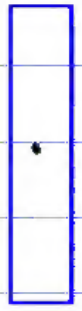
→ For a solid body, $\sum \vec{F} = m\vec{a}_G$ (also written as $m\bar{a}$)

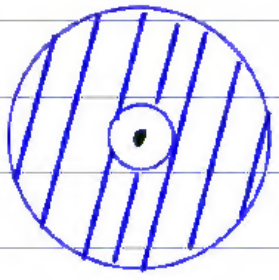
where m = total mass of body $= \int_{\text{solid body}} dm$

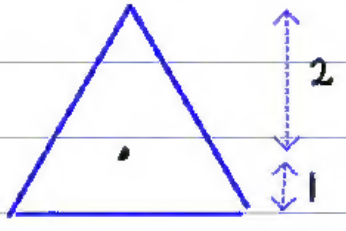
$$\vec{a}_G = \dot{\vec{V}}_G = \ddot{\vec{r}}_G$$

$$\vec{r}_G = \int_{\text{solid body}} (\vec{r}\, dm)/m$$

eg.





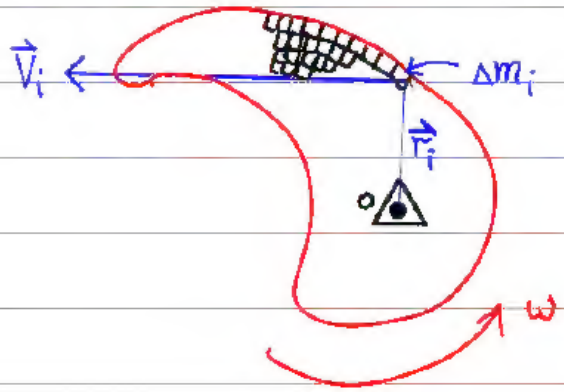


Rotation

For particles, $\dot{\vec{H}}_o = \sum \vec{M}_o$; $\dot{\vec{H}}_G = \sum \vec{M}_G$

Consider a solid body to be many particles glued together:

N particles, of mass Δm_i



$$\rightarrow \vec{H}_o = \sum_{i=1}^{N} (\Delta m_i \, \vec{r}_i \times \vec{V}_i)$$

but $\vec{V}_i \perp \vec{r}_i$; $|\vec{V}_i| = \omega r_i$

$$\rightarrow |\vec{H}_o| = \sum_{i=1}^{N} \left[(\Delta m_i)(r_i)(\omega r_i) \right]$$

$$\rightarrow |\vec{H}_o| = \omega \sum_{i=1}^{N} \Delta m_i r_i^2$$

$$\rightarrow \boxed{\vec{H}_o = \vec{\omega} I_o}, \quad \text{where } \boxed{I_o = \sum_{i=1}^{N} (r_i^2 \, \Delta m_i) \quad \text{or} \quad I_o = \int_{\text{solid body}} r^2 \, dm}$$

I_o = mass moment of inertia about o

$\frac{d}{dt} \{ \vec{H}_o = \vec{\omega} I_o \} \rightarrow \dot{\vec{H}}_o = I_o \vec{\alpha}$

but we already know $\sum \vec{M}_o = \dot{\vec{H}}_o$

$$\rightarrow \boxed{\sum \vec{M}_o = I_o \vec{\alpha}}$$

& similarly, ... $\boxed{\sum \vec{M}_G = I_G \vec{\alpha}}$

→ Key equations for planar kinetics:

$$1/ \quad \Sigma \vec{F} = m \vec{a}_G$$
$$2/ \quad \Sigma \vec{M}_O = I_O \vec{\alpha} \quad +\circlearrowleft$$
$$3/ \quad \Sigma \vec{M}_G = I_G \vec{\alpha} \quad +\circlearrowleft$$

(use either 2/ or 3/ but not both)

Note that the $\Sigma \vec{M}$ equation is valid in general only at a fixed axis O, or at G.

~~The textbook tries to combine equations 1/ and 3/ to allow α to be calculated at any point P:~~

$$\Sigma \vec{M}_P = I_G \alpha + m \vec{a}_G d$$

~~where d is based on location of P, G, and direction of $\vec{a}$.~~

Helpful guideline:

If the body is rotating about a fixed axis 'O':

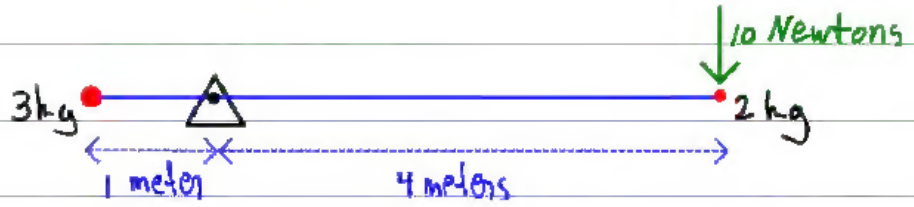
→ use equations 1/ & 2/

(It will save you some arithmetic)

If there is no fixed axis, then you have no choice:

→ use equations 1/ & 3/

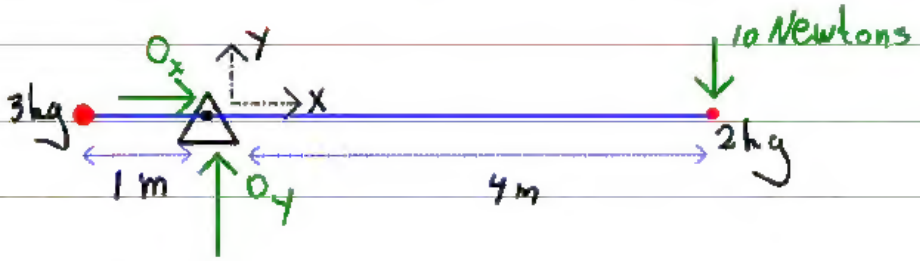
Eg. Lopsided barbell:



Given: Barbell is on a frictionless horizontal plane
Mass of bar is negligible
The two red balls can be considered as point masses.

Find α

1/ Draw FBD. Define axes & rotation +ve direction



2/ Note there is a fixed axis at O •

→ use

1/ $\sum \vec{F} = m \vec{a}_G$

2/ $\sum \vec{M}_o = I_o \vec{\alpha}$

~~3/ $\sum \vec{M}_G = I_G \vec{\alpha}$~~

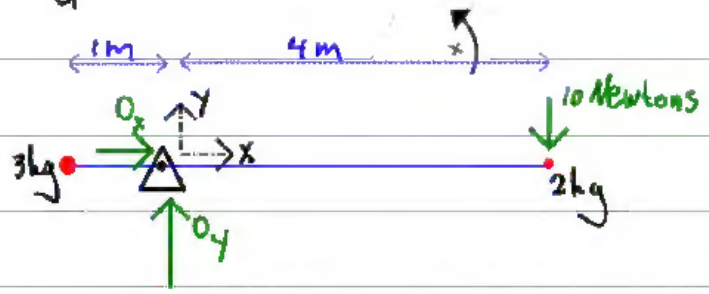
3/ $I_o = \sum_{i=1}^{2} (r_i^2 m_i) = (4\text{meters})^2(2\text{kg}) + (1\text{meter})^2(3\text{kg})$

$I_o = 35\ \text{kg}\cdot\text{m}^2$

4/ Apply Eqn 2 → $\sum \vec{M}_o = I_o \vec{\alpha}$ ($-(10)(4)$ = $\sum \vec{M}_o$, 35 = I_o)

$\alpha = -1.143\ s^{-2}$

Suppose you inadvisedly decide to use the equation $\sum \vec{M}_G = I_G \vec{\alpha}$



1/ Find $\vec{r}_G = \sum(m_i \vec{r}_i)/(\sum m_i) = [\,2kg)(4\hat{i}) + (3kg)(-\hat{i})\,]/(2kg+3kg)$

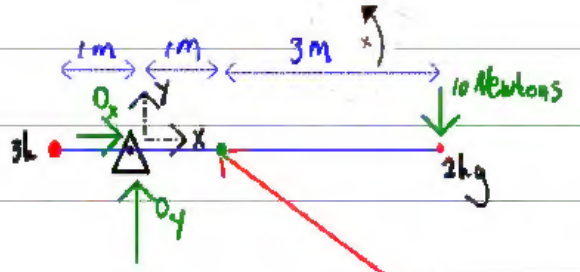
$\rightarrow \vec{r}_G = 1\,\hat{i}$ meters

2/ $I_G = \sum_{i=1}^{2}(r_i^2 m_i) = [\,(3\text{ meters})^2(2kg) + (2\text{ meters})^2(3kg)\,]$

(r_i: distance from G)

$\rightarrow I_G = 30\ kg \cdot m^2$ $\quad < I_O$

3/ Apply $\sum \vec{M}_G = I_G \vec{\alpha}$



★ $-(10\,N)(3\,m) - (O_y)(1\,m) = 30\,\alpha$

★★ 4/ Use $\sum F_y = m(a_G)_y$

$O_y - 10\,N = (5\,kg)(a_G)_y$

G: {x, y} = {1, 0}

★★★ 5/ and use $(a_G)_y = (a_G)_t = \alpha r_{OG} = (\alpha)(1m)$

Combine the equations ★, ★★, and ★★★ to solve for α, $(a_G)_y$, O_y

→ $\alpha = -1.143\ s^{-2}$

$(a_G)_y = -1.143\ m/s^2$

$O_y = 4.3$ Newtons

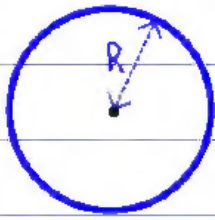
too much work!

We often need to find I_G for an engineering component:

Recall: For many particles, $I_G = \sum m_i r_i^2$

1/ I_G for a hollow ring of mass m, radius R:

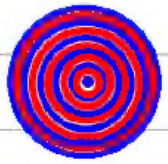
$$I_G = \sum m_i r_i^2 = R^2 \sum m_i = mR^2$$



2/ I_G for a solid disk: Divide up disk into many thin rings of radius r, width dr, thickness into the page t, density ρ

For each ring,

$$I_{G,ring} = mr^2 = (\text{volume})\rho r^2$$
$$= (2\pi r\, dr\, t)\rho r^2$$



For entire disk,

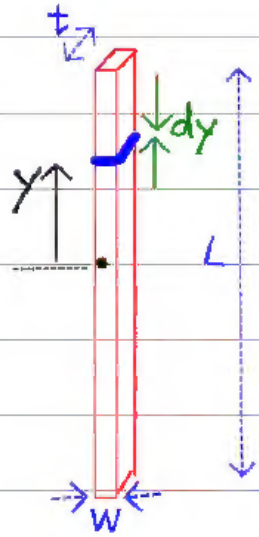
$$I_G = \int_0^R (2\pi\rho t\, r^3)\, dr = 2\pi\rho t \frac{R^4}{4}$$

but $m = \rho \pi R^2 t$ → I_G for a disk $= \frac{mR^2}{2}$

3/ I_G for a long thin stick:

consider a slab located at coordinate y

I_G for blue slab $= mr^2 = (\rho)(\text{volume})y^2$
$= (\rho)(w t\, dy)\, y^2$



→ I_G for long thin stick $= \int_{y=-\frac{L}{2}}^{L/2} \rho w t\, y^2\, dy$
$= w\rho t L^3/12$

but $m = wtL\rho$

→ I_G for long thin stick $= mL^2/12$

Summary

- 2D Kinetics:

$$1/ \quad \Sigma \vec{F} = m \vec{a}_G$$
$$2/ \quad \Sigma \vec{M}_O = I_O \vec{\alpha} \ \circlearrowleft$$
$$3/ \quad \Sigma \vec{M}_G = I_G \vec{\alpha} \ \circlearrowleft$$

- Moment of inertia I_P can be calculated for any point P, but I_G and I_O are the most useful

- $I_P = \sum m_i r_i^2$ (for individual particles)
  or $\int r^2 dm$ (for a continuous mass)

  where r is the distance from P to each bit of mass

- $I_G = mR^2$ (thin ring of radius R)
  $mR^2/2$ (solid disk of radius R)
  $mL^2/12$ (thin stick of length L)

Next Lecture

- Example problems from Meriam 6/1 – 6/5 : Plane kinetics